Interp. polynomials in Newton form. Divided differences.

Want to interpolate a function $f(x)$ at points: $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$. and obtain the formula for the interpolating polynomial.

Newton Form:

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots$$
$$\ldots + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}).$$

Determination of a_i, $i = 0, 1, .., n$.

$$p_n(x_0) = a_0 = f(x_0)$$

$$p_n(x_1) = a_0 + a_1(x_1 - x_0) = f(x_0) + a_1(x_1 - x_0) = f(x_1)$$

$$\Rightarrow \qquad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

We will introduce this notation:

$$a_0 = f(x_0) = f[x_0]. \qquad \text{Zeroth divided diffce.}$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1].$$

First divided diffce.

$$p_n(x_2) = a_0 + a_1(x_2-x_0) + a_2(x_2-x_0)(x_2-x_1) = f(x_2)$$

$$\Rightarrow \quad a_2 = \frac{f(x_2) - \overset{=f(x_0)}{a_0} - a_1(x_2-x_0)}{(x_2-x_0)(x_2-x_1)}$$

or

$$a_2 = \frac{f(x_2) - f(x_0) - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}(x_2-x_0)}{(x_2-x_0)(x_2-x_1)}$$

$$\Rightarrow \quad a_2 = \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

or $$a_2 = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2-x_0} = f[x_0,x_1,x_2].$$

Second divided diffce.

Inductively,

$$a_k = \frac{f[x_1,x_2,\dots,x_k] - f[x_0,x_1,\dots,x_{k-1}]}{x_k-x_0} =$$

$$= f[x_0,x_1,\dots,x_k]$$

kth divided difference.

$$\cancel{(x_2-x_0)}\left[\frac{f(x_2)-f(x_0)}{(x_2-x_0)(x_2-x_1)} - \frac{(f(x_1)-f(x_0))/(x_1-x_0)}{x_2-x_0}\right]$$

$$\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} - \frac{f(x_1)}{(x_1-x_0)(x_2-x_1)} - \frac{f(x_0)}{(x_2-x_0)(x_2-x_1)}$$

$$+ \frac{f(x_0)}{(x_1-x_0)(x_2-x_1)}$$

$$f(x_0)\left[\frac{1}{(x_1-x_0)(x_2-x_1)} - \frac{1}{(x_2-x_0)(x_2-x_1)}\right]$$

$$= f(x_0)\left[\frac{x_2-\cancel{x_0}-(x_1-\cancel{x_0})}{(x_2-x_0)(x_1-x_0)(x_2-x_1)}\right]$$

$$= f(x_0)\left[\frac{\cancel{x_2-x_1}}{(x_2-x_0)(x_1-x_0)\cancel{(x_2-x_1)}}\right]$$

Therefore,

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) +$$

$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots\cdots$$

$$+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

In general, we will define for any $i = 0, 1, \ldots, n$

<u>Zeroth div. diffce:</u> $f[x_i] = f(x_i)$

<u>First div. diffce:</u> $f[x_i, x_{i+1}] = \dfrac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$

<u>Second div. diffce</u>: $f[x_i, x_{i+1}, x_{i+2}] = \dfrac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$

<u>kth divided difference.</u>

$$f[x_i, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

Divided Difference Table for polynomials degree at most three.

$(x_0, f(x_0)), (x_1, f(x_1)),$
$(x_2, f(x_2)), (x_3, f(x_3))$

x_0	$f[x_0] = a_0$			
x_1	$f[x_1]$	$f[x_0, x_1] = a_1$		
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2] = a_2$	
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3] = a_3$

$$P_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) +$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

Matrix Notation.

$$
\begin{array}{llll}
x_0 & \overset{=F_{00}}{f[x_0]} & & \\
 & & \overset{=F_{11}}{f[x_0,x_1]} & \\
x_1 & \overset{=F_{10}}{f[x_1]} & & \overset{=F_{22}}{f[x_0,x_1,x_2]} \\
 & & \overset{=F_{21}}{f[x_1,x_2]} & \\
x_2 & \overset{=F_{20}}{f[x_2]} & & \overset{=F_{32}}{f[x_1,x_2,x_3]} \\
 & & \overset{=F_{31}}{f[x_2,x_3]} & \\
x_3 & \overset{=F_{30}}{f[x_3]} & &
\end{array}
\qquad \overset{=F_{33}}{f[x_0,x_1,x_2,x_3]}
$$

$i \geq j$

$$F_{ij} = f[x_{i-j}, x_{i-j+1}, \dots, x_i]$$

Algorithm

For $i = 1, 2, \dots, n$

For $j = 1, 2, \dots, i$

$$F_{ij} = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$$

end

end

output $(F_{00}, F_{11}, F_{22}, \dots, F_{nn})$

Clearly, $F_{ii} = f[x_0, x_1, \dots, x_i]$

... the Newton forward divided-difference form of the interpolatory polynomial are along the diagonal in the table. The polynomial is

$$
\begin{aligned}
P_4(x) = {} & 0.7651977 - 0.4837057(x - 1.0) - 0.1087339(x - 1.0)(x - 1.3) \\
& + 0.0658784(x - 1.0)(x - 1.3)(x - 1.6) \\
& + 0.0018251(x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9).
\end{aligned}
$$

Notice that the value $P_4(1.5) = 0.5118200$ agrees with the result in Section 3.1, Example 3, as it must because the polynomials are the same.

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, \ldots, x_i]$	$f[x_{i-4}, \ldots, x_i]$
0	1.0	0.7651977				
			−0.4837057			
1	1.3	0.6200860		−0.1087339		
			−0.5489460		0.0658784	
2	1.6	0.4554022		−0.0494433		0.0018251
			−0.5786120		0.0680685	
3	1.9	0.2818186		0.0118183		
			−0.5715210			
4	2.2	0.1103623				

$\Rightarrow F_{ij} = \dfrac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$

DEFINE $F_{i,i} \equiv f[x_0, x_1, \ldots, x_i]$ $F_{ij} \equiv f[x_{i-j}, x_{i-j+1}, x_i]$ $i \geq j.$

Example: Interpolating polynomial through the points: $(-1,10), (0,3), (1,6), (2,3)$.

x_i's	$f(x_i$'s$)$	1st D.D.	2nd D.D.	3rd D.D.
-1	$\boxed{10}$			
		$\frac{3-10}{0-(-1)} = \boxed{-7}$		
0	3		$\frac{3+7}{1-(-1)} = \boxed{5}$	
		$\frac{6-3}{1-0} = 3$		$\frac{-3-5}{2-(-1)} = \boxed{-\frac{8}{3}}$
1	6		$\frac{-3-3}{2-0} = -3$	
		$\frac{3-6}{2-1} = -3$		
2	3			

$$\therefore \boxed{P_3(x) = 10 - 7(x+1) + 5(x+1)x - \frac{8}{3}(x+1)x(x-1)}$$